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Slow motion of two spheres in a shear field

By C. J. LIN, K. J. LEE AND N. F. SATHER
Department of Chemical Engineering, University of Washington, Seattle, Washington 98105

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The exact solution of the Stokes equations for the creeping motion of two spheres of arbitrary size and arbitrarily oriented with respect to a shear field is obtained by use of spherical bipolar co-ordinates. Numerical results are given for two special cases: (1) the free motion of two equal-sized spheres in simple shear flow and (2) the free motion of a sphere near a wall in the rotational shear field between two parallel disks rotating at different rates. The sphere trajectories calculated for the first of these problems are found to agree fairly well with those observed experimentally.

1. Introduction

In this paper we report some results obtained from the exact solution of the Stokes equations for the quasi-static motion of two spherical particles in the shear field of a viscous fluid. Since the method used to obtain the solution can accommodate spheres of different size, a description of the limiting case of the motion of a particle near a planar boundary is also obtained. Our particular interest in this problem derives from its application in the theory of the viscosity of moderately concentrated suspensions of spherical particles. There are, however, other flow situations for which the analysis described here has potential application, for example, the settling of particles in a shear field and the motion of particles near walls.

The solution is obtained by an extension of an exact procedure employing spherical bipolar co-ordinates that has been developed by several previous investigators. The first applications of the bipolar co-ordinate method were to the class of axisymmetric flows associated with the translation and rotation of spheres in a quiescent fluid. Jeffery (1915) analyzed the flow in the vicinity of two spheres rotating slowly about their line-of-centres and the corresponding problem of the rotation of a sphere about an axis perpendicular to a plane boundary. Stimson & Jeffery (1926) considered the creeping motion of two spheres moving without rotation along their line-of-centres. The quasi-steady translation of a sphere perpendicular to a wall was determined independently by Brenner (1961) and Maude (1961). The application of the method to non-axisymmetric motions was initiated by Dean & O'Neill (1963), who studied the rotation of a sphere about an axis parallel to a wall. The problem of the translation of a sphere parallel to a wall was solved by O'Neill (1964). Because of the linearity of the Stokes equations the results for simple motions can be combined to generate solutions of more complex cases. In this manner Goldman, Cox & Brenner (1966) determined the motion of two arbitrarily oriented spheres through a quiescent fluid, and Wakiya (1967) considered the more general problem of two spheres moving in a non-uniform velocity field. However, Wakiya's results are for overall flows having a plane of symmetry and so do not describe the sphere and fluid motions when the particles are oriented arbitrarily with respect to a shear field. The solution of the latter problem is reported here. Since many of the steps in the analysis are similar to those of the above papers, only a brief summary of the method will be given; a complete discussion of the method and results is given in the thesis by Lin (1968).

The formal solution of the problem is described in §2, leaving for §§3 and 4 the discussion of numerical results for specific examples. Two limiting cases are considered there: the free motion of two neutrally buoyant, equal-sized spheres in simple shear flow and the free motion of a neutrally buoyant sphere near a wall in the rotational shear field between two parallel disks that are rotating with different angular velocities.

2. Analysis for spheres of arbitrary size

2.1. *Velocity field*

Consider the problem of determining the quasi-steady velocity field in the vicinity of two spheres moving with arbitrary velocities in the shear field of an incompressible Newtonian fluid. We denote by $\mathbf{u}$ and p the local velocity and pressure fields, and by $\mathbf{u}_0$ and p_0 the corresponding fields that would be obtained were the spheres not there. The motion is assumed to be sufficiently slow that $\mathbf{u}$ and $\mathbf{u}_0$ satisfy the Stokes equations for creeping flow, and the undisturbed fields $\mathbf{u}_0$ and p_0 are presumed known. The relative velocity and pressure fields, $\mathbf{v} = \mathbf{u}-\mathbf{u}_0$ and $q = p-p_0$, are then the solutions of

$$\nabla . \mathbf{v} = 0 \tag{1}$$

and

$$\mu\nabla^2\mathbf{v} = \nabla q \tag{2}$$

that satisfy both the asymptotic conditions $\mathbf{v} \to 0$ and $q \to 0$ far from the spheres and the no-slip boundary conditions on the sphere surfaces. The geometric quantities needed for stating the boundary conditions are shown in figure 1. The spheres are denoted I and II, their radii by a_{I} and a_{II}, and the distance between their centres is d. The bispherical geometry of the boundary conditions is most easily accommodated by using the spherical bipolar co-ordinate system (ξ, η, ϕ) which moves with the spheres. The origin of the system is located on the line-of-centres at the point which divides d into unequal parts h_{I} and h_{II} such that $h_{\mathrm{I}}^2-a_{\mathrm{I}}^2 = h_{\mathrm{II}}^2-a_{\mathrm{II}}^2 = c^2$, where

$$c = \tfrac{1}{2}d^{-1}\{[d^2-(a_{\mathrm{I}}^2+a_{\mathrm{II}}^2)]^2-4a_{\mathrm{I}}^2a_{\mathrm{II}}^2\}^{\frac{1}{2}}.$$

The co-ordinates themselves are defined by

$$\rho = \frac{c\sin\eta}{\cosh\xi-\cos\eta}, \quad z = \frac{c\sinh\xi}{\cosh\xi-\cos\eta}$$

and $\phi = \phi$, where ρ, z and ϕ are the co-ordinates in the cylindrical system oriented so that the z axis coincides at all times with the line-of-centres. In bispherical co-ordinates the surface of sphere I is given by $\xi = \alpha_I$ and that of sphere II by $\xi = -\alpha_{II}$, where the α_N are constants

$$\alpha_N = \operatorname{cosech}^{-1}(a_N/c) = \cosh^{-1}(h_N/a_N).$$

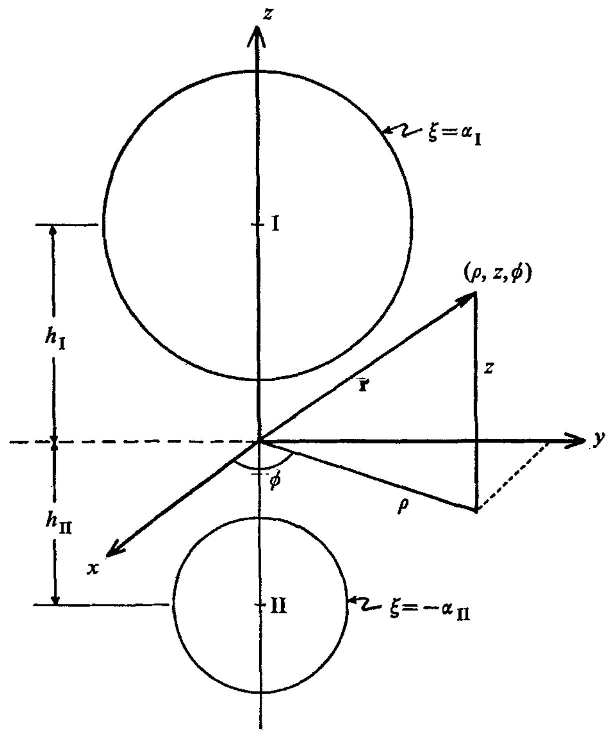

FIGURE 1. Co-ordinate geometry. The surfaces of the spheres are defined by specifying the value of the bispherical co-ordinate ξ: for sphere I, $\xi = \alpha_I$ and for sphere II, $\xi = -\alpha_{II}$, where $\alpha_N = \cosh^{-1}(h_N/a_N) = \operatorname{cosech}^{-1}(a_N/c)$.

With these definitions the boundary conditions are

$$\mathbf{v}(\mathbf{r}_I) = \mathbf{V}^I \equiv \mathbf{U}^I + \mathbf{\Omega}^I \times (\mathbf{r}_I - h_I \mathbf{e}_z) - \mathbf{u}_0^I \quad \text{at} \quad \xi = \alpha_I \tag{3a}$$

and $$\mathbf{v}(\mathbf{r}_{II}) = \mathbf{V}^{II} \equiv \mathbf{U}^{II} + \mathbf{\Omega}^{II} \times (\mathbf{r}_{II} + h_{II} \mathbf{e}_z) - \mathbf{u}_0^{II} \quad \text{at} \quad \xi = -\alpha_{II}. \tag{3b}$$

Here $\mathbf{U}^N$ and $\mathbf{\Omega}^N$ are the translational and rotational velocities of the spheres, $\mathbf{r}_N$ is the vector from the origin to a point on the surface of sphere N, and $\mathbf{u}_0^N$ is the value of $\mathbf{u}_0$ at $\mathbf{r}_N$.

From (1) and (2) we know that the pressure field satisfies the Laplace equation while the velocity field is of the form $\mathbf{v} = \mathbf{v}^P + \mathbf{v}^H$, where $\mathbf{v}^P = (q/2\mu)\,\mathbf{r}$ is a particular solution of (2) and $\mathbf{v}^H$ also satisfies the Laplace equation. The general solution of the Laplace equation in spherical bipolar co-ordinates can be obtained by separation of variables (Morse & Feshbach 1953), giving the following expressions for the pressure field and the cylindrical components of the velocity field

$$q = (\mu/c) \sum_{k=0}^{\infty} (\overline{W}_k^0 \sin k\phi + \overline{W}_{-k}^0 \cos k\phi), \tag{4a}$$

$$v_\rho = \frac{1}{2}\sum_{k=0}^{\infty}\{((\rho/c)\,\overline{W}_k^0 + W_k^{+1} + W_k^{-1})\sin k\phi + ((\rho/c)\,\overline{W}_{-k}^0 + W_{-k}^{+1} + W_{-k}^{-1})\cos k\phi\}, \quad (4b)$$

$$v_\phi = \frac{1}{2}\sum_{k=0}^{\infty}\{(W_k^{-1} - W_k^{+1})\cos k\phi - (W_{-k}^{-1} - W_{-k}^{+1})\sin k\phi\} \quad (4c)$$

and
$$v_z = \frac{1}{2}\sum_{k=0}^{\infty}\{((z/c)\,\overline{W}_k^0 + 2W_k^0)\sin k\phi + ((z/c)\,\overline{W}_{-k}^0 + 2W_{-k}^0)\cos k\phi\}, \quad (4d)$$

where

$$W_m^i = (\cosh\xi - \cos\eta)^{\frac{1}{2}} \sum_{n=|m|+i}^{\infty} [A_{mn}^i \cosh(n+\tfrac{1}{2})\,\xi + B_{mn}^i \sinh(n+\tfrac{1}{2})\,\xi]\, P_n^{|m|+i}(\cos\eta) \quad (5)$$

for $-\infty < m < \infty$. Here $P_n^{|m|}(\cos\eta)$ is the associated Legendre polynomial of the first kind of order n and rank $|m|$, and $\overline{W}_m^0$ is of the form (5), but with A_{mn}^0 and B_{mn}^0 replaced by $\bar{A}_{mn}^0$ and $\bar{B}_{mn}^0$.

The unspecified constants A_{mn}^i, B_{mn}^i, $\bar{A}_{mn}^0$ and $\bar{B}_{mn}^0$ are determined by requiring that the solution satisfy the continuity equation (1) and the boundary conditions (3); details of the procedure are given by Lin (1968).† This completes the formal solution of the flow problem.

2.2. *Forces and torques on the spheres*

The hydrodynamical forces and torques experienced by the spheres during the course of their motion also can be expressed in terms of the coefficients A_{mn}^i and B_{mn}^i of the velocity and pressure fields. Considering first sphere I, we define a spherical co-ordinate system (R, Θ, Φ) having its origin at the centre of sphere I and its orientation such that its polar direction is the z axis and Φ is the azimuthal angle ϕ of the original cylindrical co-ordinate system. The transformation relations between the two systems are then $R = \{\rho^2 + (z-h_{\mathrm{I}})^2\}^{\frac{1}{2}}$, $\Theta = \tan^{-1}\rho/(z-h_{\mathrm{I}})$ and $\Phi = \phi$.

The force and torque on sphere I are defined by

$$\mathbf{F} = a_{\mathrm{I}}^2 \int_0^{2\pi}\int_0^{\pi} (\mathbf{p}_R)_{R=a_{\mathrm{I}}} \sin\Theta\, d\Theta\, d\Phi \quad (6a)$$

and
$$\mathbf{T} = a_{\mathrm{I}}^2 \int_0^{2\pi}\int_0^{\pi} (\mathbf{R}\times\mathbf{p}_R)_{R=a_{\mathrm{I}}} \sin\Theta\, d\Theta\, d\Phi, \quad (6b)$$

where $\mathbf{p}_R$ is the radial component of the fluid stress $\mathfrak{p}$:

$$\mathbf{p}_R = \mathbf{e}_R \cdot \mathfrak{p} = -\mathbf{e}_R p + \mu\left(\frac{\partial \mathbf{u}}{\partial R} - \frac{\mathbf{u}}{R}\right) + \frac{\mu}{R}\nabla(\mathbf{R}\,.\,\mathbf{u}). \quad (7)$$

Evaluation of the surface integrals in (6) using the bispherical-co-ordinate solutions for v_ρ, v_ϕ and v_z gives after lengthy calculation the following expressions for the Cartesian components of $\mathbf{F}$ and $\mathbf{T}$:

$$F_x = -2^{\frac{3}{2}}\pi\mu c \sum_{n=0}^{\infty} (A_{-1n}^{-1} \pm B_{-1n}^{-1}), \quad (8a)$$

† Copies of a document containing an account of the derivation of the solution and a tabulation of the resulting formulae for the constants can be obtained on request from the Editor or from the authors.

$$F_y = -2^{\frac{3}{2}}\pi\mu c \sum_{n=0}^{\infty} (A_{1n}^{-1} \pm B_{1n}^{-1}), \tag{8b}$$

$$F_z = -2^{\frac{5}{2}}\pi\mu c \sum_{n=0}^{\infty} (A_{0n}^{0} \pm B_{0n}^{0}), \tag{8c}$$

$$T_x = -2^{\frac{3}{2}}\pi\mu c^2 \sum_{n=0}^{\infty} \{2n+1-(h_N/c)\}(B_{1n}^{-1} \pm A_{1n}^{-1}), \tag{8d}$$

$$T_y = -2^{\frac{3}{2}}\pi\mu c^2 \sum_{n=0}^{\infty} \{2n+1-(h_N/c)\}(B_{-1n}^{-1} \pm A_{-1n}^{-1}), \tag{8e}$$

and

$$T_z = 2^{\frac{3}{2}}\pi\mu c^2 \sum_{n=0}^{\infty} \{(A_{0n}^{-1} \pm B_{0n}^{-1}) + n(n+1)(A_{0n}^{+1} \pm B_{0n}^{+1})\}. \tag{8f}$$

In each case the upper sign $(+)$ is to be used for sphere I, while the lower sign $(-)$ is for sphere II.

The different fluid-particle motions that can be described by the foregoing analysis can be separated into two types: those for which the forces and torques are to be inferred from observations of the sphere motions and those for which the instantaneous linear and angular velocities of the spheres are to be calculated from knowledge of the forces and torques. In the former case the right-hand sides of (8*a*–*f*) are completely known so the forces and torques can be computed directly. An example of a problem of this type is the determination of the forces and torques needed to hold the spheres fixed in a given configuration relative to the overall flow. In the latter case the left-hand sides of (8*a*–*f*) are known, so the velocities $\mathbf{U}^N$ and $\mathbf{\Omega}^N$, which appear implicitly in the right-hand sides, can be calculated. Included in this category is the calculation of settling trajectories and velocities of pairs of interacting spheres. Another example of a problem of this type is the determination of the trajectories of neutrally buoyant spheres moving freely (no net forces and torques) through a fluid. The results of calculations of sphere trajectories for two different flow geometries are described in the remainder of this paper.

3. Trajectories of two spheres in a simple shear field

One of the more useful results that can be obtained from the preceding analysis is the calculation of interactional trajectories of pairs of neutrally buoyant spheres in a shear field. To be specific, we assume the spheres to be of equal size ($a_{\mathrm{I}} = a_{\mathrm{II}} = a$) and the undisturbed field $\mathbf{u}_0$ to be a simple shearing motion. The trajectories for unequal-sized spheres and for other choices for $\mathbf{u}_0$, although not reported here, can also be obtained from the analysis. Two simplifications result from these assumptions: because $\mathbf{u}_0$ is linear in $\mathbf{r}$ all of the coefficients A_{mn}^i and B_{mn}^i for $|m| > 2$ are zero, and because the spheres are identical in size the non-vanishing coefficients depend upon only one scalar quantity, $\alpha = \cosh^{-1}(h/a)$, where $h = h_{\mathrm{I}} = h_{\mathrm{II}} = \frac{1}{2}d$.

The particle trajectories that we wish to describe are the time tracings of the relative positions of the two spheres with respect to the field $\mathbf{u}_0$. Hence, we shall adopt a second set of Cartesian co-ordinates (x', y', z') having its origin at the centre of sphere II and oriented such that its x' axis is parallel to the direction of the undisturbed velocity, that is, $\mathbf{u}_0 = \kappa \mathbf{e}_{x'} y'$, where κ is the constant shear rate. The co-ordinates x', y', z' of the centre of sphere I are made dimensionless by scaling them in units of sphere radii a. Since the spheres are neutrally buoyant and inertial effects are presumed negligible, the hydrodynamical forces and torques acting on the spheres must vanish:

$$\mathbf{F}^{\mathrm{I}} = \mathbf{F}^{\mathrm{II}} = \mathbf{T}^{\mathrm{I}} = \mathbf{T}^{\mathrm{II}} = 0. \tag{9}$$

From these equations we find that $\mathbf{U}^{\mathrm{I}} = -\mathbf{U}^{\mathrm{II}} = \mathbf{U}$ and $\mathbf{\Omega}^{\mathrm{I}} = \mathbf{\Omega}^{\mathrm{II}} = \mathbf{\Omega}$, and we obtain, in addition, expressions for $\mathbf{U}$ and $\mathbf{\Omega}$ in terms of the A^i_{mn} and B^i_{mn}.

The expressions so obtained for the dimensionless velocities $\mathbf{U}^* = \mathbf{U}/(\frac{1}{2}a\kappa)$ and $\mathbf{\Omega}^* = \mathbf{\Omega}/(\frac{1}{2}\kappa)$ can be written

$$\left.\begin{aligned} U^*_x &= \mathscr{A}(r')\cos^2\phi'\sin\theta' - \mathscr{B}(r')\sin^2\phi'\sin\theta',\\ U^*_y &= -\{\mathscr{A}(r')+\mathscr{B}(r')\}\sin\phi'\cos\phi'\sin\theta'\cos\theta',\\ U^*_z &= \mathscr{C}(r')\sin\phi'\cos\phi'\sin^2\theta',\\ \Omega^*_x &= \{\mathscr{D}(r')+\mathscr{E}(r')\}\sin\phi'\cos\phi'\sin\theta'\cos\theta',\\ \Omega^*_y &= \mathscr{D}(r')\cos^2\phi'\sin\theta' - \mathscr{E}(r')\sin^2\phi'\sin\theta',\\ \text{and}\quad \Omega^*_z &= -\cos\theta', \end{aligned}\right\} \tag{10}$$

where θ', ϕ' and $r' = 2h/a = 2\cosh\alpha$ are the spherical co-ordinates of the centre of sphere I in the space-fixed reference frame. Computed values of the functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ are given in table 1 for a number of values of r'.

The trajectories of sphere I relative to sphere II are the integrals of the dynamical equations

$$dr'/dt^* = U^*_z,$$

$$d\theta'/dt^* = -U^*_y/r',$$

and

$$d\phi'/dt^* = U^*_x/r'\sin\theta',$$

where $t^* = \kappa t$ is the dimensionless time and U^*_x, U^*_y and U^*_z are given in (10). These give

$$\cos\theta' = Bf(r') \tag{11a}$$

and

$$\sin\theta'\sin\phi' = \pm f(r')\{C+h(r')\}^{\frac{1}{2}}, \tag{11b}$$

where

$$f(r') = \exp\left\{-\int\frac{\mathscr{A}+\mathscr{B}}{r'\mathscr{C}}\,dr'\right\} \tag{12a}$$

and

$$h(r') = \int\frac{2\mathscr{A}}{r'\mathscr{C}f^2}\,dr' \tag{12b}$$

and B and $C \geqslant 0$ are constants. The trajectories of sphere I are then the lines of intersection of the two families of surfaces defined by (11). The y' and z' co-ordinates of the trajectories can be calculated as functions of r' from

$$y' = \pm r'f(r')\{C+h(r')\}^{\frac{1}{2}} \tag{13a}$$

and

$$z' = Br'f(r'), \tag{13b}$$

using the values of $f(r')$ and $h(r')$ given in table 1. The constants B and C in (13)

α	r'	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{C}$	$\mathscr{D}$	$\mathscr{E}$	f	h
3·00	20·1353	−0·0000	20·1353	20·1230	−0·9997	1·0003	0·0497	0·0000
2·40	11·1139	−0·0002	11·1137	11·0738	−0·9982	1·0018	0·0901	0·0011
1·80	6·2149	−0·0018	6·2132	6·0882	−0·9897	1·0103	0·1612	0·0071
1·50	4·7048	−0·0055	4·6994	4·4847	−0·9765	1·0234	0·2149	0·0168
1·20	3·6213	−0·0156	3·6057	3·2471	−0·9499	1·0501	0·2858	0·0381
1·10	3·3370	−0·0217	3·3153	2·8928	−0·9366	1·0633	0·3132	0·0491
1·00	3·0862	−0·0298	3·0564	2·5603	−0·9209	1·0791	0·3429	0·0628
0·90	2·8662	−0·0402	2·8260	2·2451	−0·9024	1·0976	0·3750	0·0794
0·80	2·6749	−0·0533	2·6216	1·9433	−0·8810	1·1190	0·4096	0·0994
0·70	2·5103	−0·0694	2·4409	1·6507	−0·8567	1·1433	0·4470	0·1230
0·60	2·3709	−0·0887	2·2822	1·3637	−0·8291	1·1709	0·4874	0·1504
0·50	2·2553	−0·1114	2·1438	1·0793	−0·7981	1·2019	0·5315	0·1819
0·40	2·1621	−0·1378	2·0244	0·7971	−0·7631	1·2369	0·5805	0·2182
0·30	2·0907	−0·1681	1·9225	0·5217	−0·7233	1·2767	0·6373	0·2605
0·20	2·0401	−0·2036	1·8366	0·2694	−0·6770	1·3230	0·7092	0·3125
0·10	2·0100	−0·2473	1·7627	0·0765	−0·6194	1·3806	0·8232	0·3860
0·05	2·0025	−0·2765	1·7260	0·0200	−0·5805	1·4195	0·9382	0·4461
0·025	2·0006	−0·2969	1·7037	0·0051	−0·5531	1·4469	1·0623	0·4962
0·010	2·0001	−0·4426	1·5575	0·0008	−0·6735	1·3265	1·2418	0·5576
0·0075	2·00006	−0·7266	1·2734	0·0005	−0·9593	1·0407	1·2748	0·5842

TABLE 1. Values of functions

are related to y'_∞ and z'_∞, the asymptotic values of y' and z' as $r' \to \infty$, by $B = z'_\infty$ and $C = (y'_\infty)^2$.

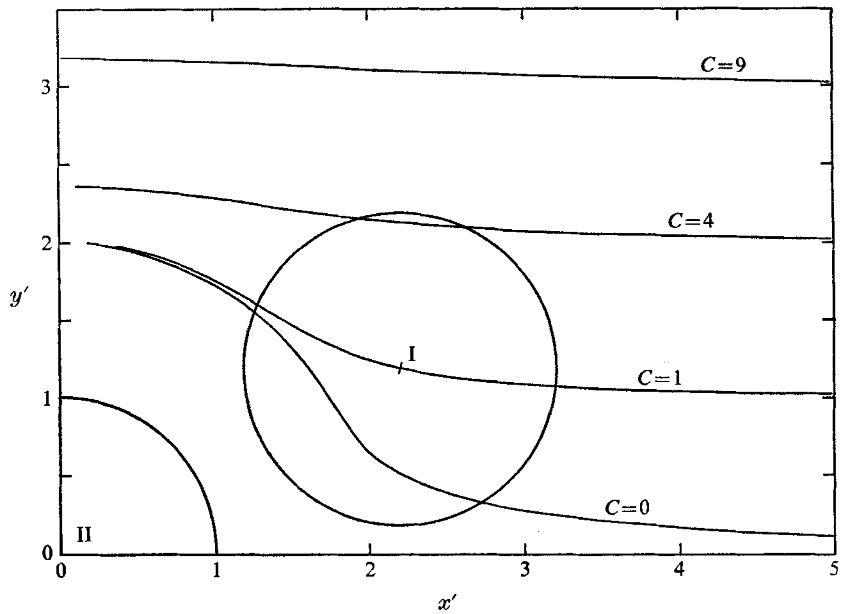

FIGURE 2. Trajectories of sphere I around sphere II in the plane $z' = 0$.

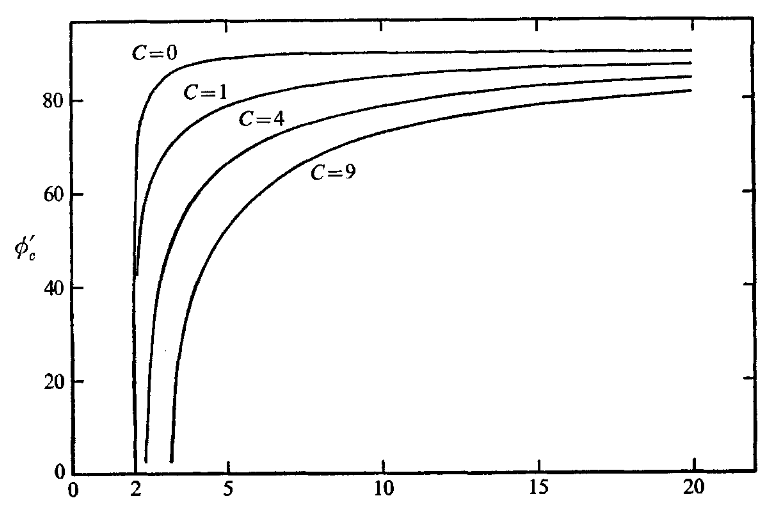

FIGURE 3. Polar plot of sphere trajectories in the plane $\theta' = \frac{1}{2}\pi$. The ordinate is $\phi'_c = \frac{1}{2}\pi - \phi'$.

As a graphical illustration of the results consider the case when $z'_\infty = 0$, for which $z'(r') = 0$ so the trajectories are two-dimensional. In figure 2 are shown the trajectories, y' *versus* x', for four values of C corresponding to asymptotic values of y' of 0, 1, 2, and 3. Although not shown, the trajectories are symmetric about the y' axis. A polar plot of these same trajectories is given in figure 3, which shows how r' varies with $\phi'_c \equiv \frac{1}{2}\pi - \phi'$ when $\theta' = \frac{1}{2}\pi$. A comparison of calculated and observed trajectories is shown in figure 4. The experimental points are taken from a figure (number 6) of a paper by Darabaner & Mason (1967). Although there are not sufficient data given to determine the precise value of C for the observed trajectory, the data fit fairly well the theoretical trajectory for $C = 2$. Since the calculated trajectory is exact, the deviations of the experimental data from it are due apparently either to errors in measurement or to effects from inertial or external forces or wall interference not included in the analysis.

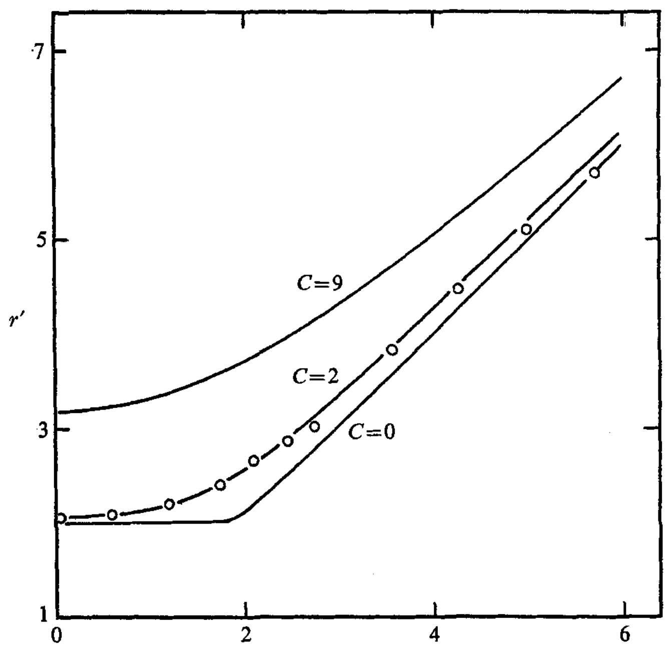

FIGURE 4. Comparison of calculated and observed trajectories in the plane $z' = 0$. The circles are experimental points from figure 6 of the paper by Darabaner & Mason (1967); the co-ordinates r' and x' are related to ρ_1 and ρ_2 of that paper by $r' = 2\rho_1/a$ and $x' = 2\rho_2/a$.

An important application of these results is the determination of the viscosity of suspensions of spherical particles having concentrations in the range of 1 % to 20 % volume fraction solids. For these concentrations the dominant dynamical event is the motion of isolated pairs of interacting spheres in a uniform shear field. It can be shown that viscosity of a suspension is given by an integral expression consisting of an integration of the local shear stress over the sphere surfaces for a given configuration of spheres and a statistical averaging with respect to the probability distribution of configurations. The surface stress can be calculated directly from the velocity field given by (4), and the configuration probability can be obtained from the trajectory equations (13). The derivation of the expression for the viscosity and numerical results calculated from it will be reported in a later communication.

4. A sphere near a plane wall

Another type of flow problem that can be treated by the analysis of §2 is the creeping motion of a sphere near a plane wall. The solution of this problem can be obtained easily by allowing the radius of one of the spheres to become infinite, i.e. $a_{\mathrm{II}} \rightarrow \infty$, while the size of the other remains fixed ($a_{\mathrm{I}} = a$). The determination of the coefficients A^i_{mn} and B^i_{mn} is simplified in this case by the fact that all of the A^0_{mn} equal zero. This follows from (4*d*) and (5) and the condition that $v_z = 0$ on the plane $z = 0$ (or $\xi = 0$).

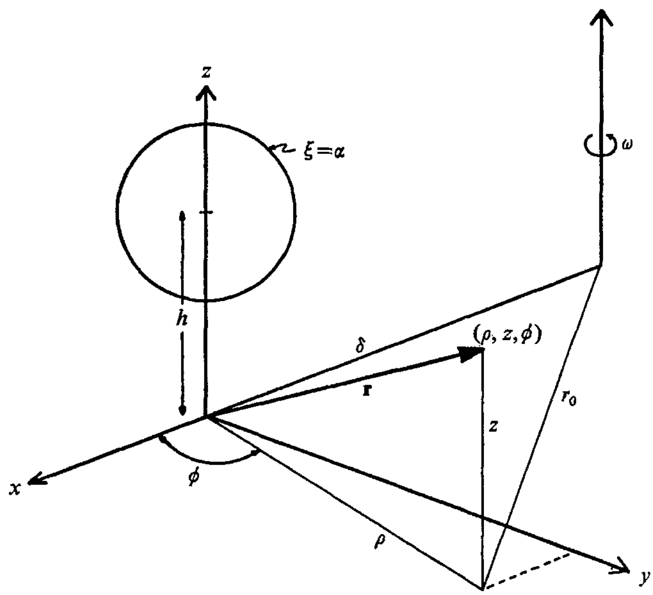

FIGURE 5. Co-ordinate geometry. The x, y plane is the plane of the stationary disk at $\xi = 0$.

As an example of a problem of this type, consider the motion of a sphere in a fluid between two parallel disks, one of which rotates with angular velocity ω while the other at $z = 0$ is stationary. The disks are assumed large enough that edge effects are negligible, and the sphere is much nearer the stationary disk so that its motion is unaffected by the rotating disk. The undisturbed velocity field is $\mathbf{u}_0 = (\omega z/l)\, \mathbf{e}_z \times \mathbf{r}_0$ where l is the distance between the disks and

$$\mathbf{r}_0 = (\rho \cos\phi + \delta)\, \mathbf{e}_x + \rho \sin\phi\, \mathbf{e}_y$$

is the position vector in the plane $z = 0$ measured from the axis of rotation (see figure 5). The boundary conditions (3) for this motion are, in cylindrical co-ordinates

$$V^{\mathrm{I}}_\rho = [U_x + \Omega_y(z-h)] \cos\phi + [U_y - \Omega_x(z-h) - (\omega\delta/l)\, z] \sin\phi,$$

$$V^{\mathrm{I}}_\phi = -[U_x + \Omega_y(z-h)] \sin\phi + \rho(\Omega_z - \omega z/l) + [U_y - \Omega_x(z-h) - (\omega\delta/l)\, z] \cos\phi,$$

and

$$V^{\mathrm{I}}_z = -\Omega_y \rho \cos\phi + U_z + \rho\Omega_x \sin\phi$$

on the surface of the sphere ($\xi = \alpha$) and $V_\rho^{\mathrm{II}} = V_\phi^{\mathrm{II}} = V_z^{\mathrm{II}} = 0$ on the plane $\xi = 0$. From these the constants A_{mn}^i and B_{mn}^i can be determined by the procedure described earlier. Note that all of the B_{mn}^0 for $|m| \geqslant 2$ are zero.

By the linearity of the Stokes equations the velocity field and hence the force and torque on the sphere each can be separated into contributions resulting from the translational, rotational and shear components of the fluid motion corresponding to the following sets of boundary conditions:

	Translation	Rotation	Shear
$\xi = \alpha$	$\mathbf{u}^t = \mathbf{U}$	$\mathbf{u}^r = \mathbf{\Omega} \times (\mathbf{r} - h\mathbf{e}_z)$	$\mathbf{u}^s = 0$
$\xi = 0$	$\mathbf{u}^t = 0$	$\mathbf{u}^r = 0$	$\mathbf{u}^s = 0$
$r \to \infty$	$\mathbf{u}^t = 0$	$\mathbf{u}^r = 0$	$\mathbf{u}^s = \mathbf{u}_0$

The force and torque on the sphere are then sums of contributions arising from these three independent flow fields. The values of the Cartesian components of these quantities calculated from (8) are given in table 2 in terms of the following dimensionless quantities.

From $m = -1$:

$$F_x^{t*} = F_x^t/6\pi\mu a U_x, \quad T_y^{t*} = T_y^t/8\pi\mu a^2 U_x,$$
$$F_x^{r*} = F_x^r/6\pi\mu a^2 \Omega_y, \quad T_y^{r*} = T_y^r/8\pi\mu a^3 \Omega_y,$$
$$F_x^s = T_y^s = 0.$$

From $m = 0$:

$$F_z^{t*} = F_z^t/6\pi\mu a U_z, \quad T_z^{r*} = T_z^r/8\pi\mu a^3 \Omega_z,$$
$$T_z^{s*} = T_z^s/8\pi\mu a^3 \omega(h/l),$$
$$F_z^r = F_z^s = T_z^t = 0.$$

From $m = +1$:

$$F_y^{t*} = F_y^t/6\pi\mu a U_y, \quad T_x^{t*} = T_x^t/8\pi\mu a^2 U_y,$$
$$F_y^{r*} = F_y^r/6\pi\mu a^2 \Omega_x, \quad T_x^{r*} = T_x^r/8\pi\mu a^3 \Omega_x,$$
$$F_y^{s*} = F_y^s/6\pi\mu a \kappa h, \quad T_x^{s*} = T_x^s/4\pi\mu a^3 \kappa,$$

where $\kappa = \omega\delta/l$ is the local shear rate in the undisturbed flow $\mathbf{u}_0$. Most of the values in table 2 have been obtained previously by other investigators, although values for $\alpha < 0{\cdot}08$ generally have not been reported. Results for F_x^{t*} and T_y^{t*} were first obtained by O'Neill (1964) and values of F_x^{r*} and T_y^{r*} by Dean & O'Neill (1963); more accurate values have since been reported by Goldman, Cox & Brenner (1967*a*). Values of F_y^{s*} and T_x^{s*} have been calculated by Goldman, Cox & Brenner (1967*b*) by a method which does not require the determination of the fluid velocity field. Numerical results for T_z^{r*} have been reported by Jeffery (1915) and for F_z^{t*} by Brenner (1961) and Maude (1961). Hence, the only new quantity needed for this flow problem is T_z^{s*}.

Two experiments that are described by this analysis are the determination of the force and torque needed to hold the sphere stationary and the determination of the instantaneous velocity of the sphere in free motion. For the former $\mathbf{U} = \mathbf{\Omega} = 0$, so

$$\mathbf{F} = \mathbf{F}^s = (6\pi\mu a\kappa h F_y^{s*})\,\mathbf{e}_y$$

and

$$\mathbf{T} = \mathbf{T}^s = 8\pi\mu a^3\{(\tfrac{1}{2}\kappa T_x^{s*})\,\mathbf{e}_x + (\omega h T_z^{s*}/l)\,\mathbf{e}_z\}.$$

α	h/a	$m = -1$				$m = 0$			$m = 1$†	
		F^{t*}_x	F^{r*}_x	T^{t*}_y	T^{r*}_y	F^{t*}_z	T^{r*}_z	T^{s*}_z	F^{s*}_y	T^{s*}_x
5·0	74·2099	−1·0076	$4{\cdot}1006 \times 10^{-9}$	$3{\cdot}0754 \times 10^{-9}$	−1·0000	1·0154	−1·0000	1·0000	1·0076	1·0000
3·0	10·0677	−1·0591	$1{\cdot}1699 \times 10^{-5}$	$8{\cdot}7745 \times 10^{-6}$	−1·0003	1·1252	−1·0001	1·0001	1·0587	0·99983
2·0	3·7622	−1·1738	$5{\cdot}6214 \times 10^{-4}$	$4{\cdot}2160 \times 10^{-4}$	−1·0059	1·4129	−1·0024	1·0023	1·1671	0·99710
1·5	2·3524	−1·3079	$3{\cdot}5231 \times 10^{-3}$	$2{\cdot}6423 \times 10^{-3}$	−1·0250	1·8375	−1·0097	1·0088	1·2780	0·99012
1·0	1·5431	−1·5675	$1{\cdot}9532 \times 10^{-2}$	$1{\cdot}4649 \times 10^{-2}$	−1·0998	3·0361	−1·0357	1·0282	1·4391	0·97423
0·7	1·2552	−1·8507	$5{\cdot}1601 \times 10^{-2}$	$3{\cdot}8701 \times 10^{-2}$	−1·2241	5·2211	−1·0708	1·0479	1·5488	0·96158
0·5	1·1276	−2·1515	$9{\cdot}8291 \times 10^{-2}$	$7{\cdot}3718 \times 10^{-2}$	−1·3877	9·2518	−1·1056	1·0622	1·6159	0·95373
0·3	1·0453	−2·6475	$1{\cdot}9403 \times 10^{-1}$	$1{\cdot}4552 \times 10^{-1}$	−1·6996	23·660	−1·1485	1·0743	1·6681	0·94769
0·1	1·00500	−3·7863	$4{\cdot}5582 \times 10^{-1}$	$3{\cdot}4187 \times 10^{-1}$	−2·5056	201·87	−1·1910	1·0814	1·6969	0·94441
0·08	1·00320	−4·0223	$5{\cdot}1326 \times 10^{-1}$	$3{\cdot}8494 \times 10^{-1}$	−2·6793	314·45	−1·1943	1·0817	1·6982	0·94426
0·05	1·00125	−4·5213	$6{\cdot}3607 \times 10^{-1}$	$4{\cdot}7705 \times 10^{-1}$	−3·0495	812·14	−1·1984	1·0821	1·6997	0·94409
0·02	1·00020	−5·4972	$8{\cdot}7877 \times 10^{-1}$	$6{\cdot}5909 \times 10^{-1}$	−3·7787	5002·5	−1·2013	1·0823	1·7004	0·9440
0·0	1·0								1·7009‡	0·943993‡

† The other quantities obtained from $m = 1$ are $F^{t*}_y = F^{t*}_x$, $F^{r*}_y = -F^{r*}_x$, $T^{t*}_x = T^{t*}_y$ and $T^{r*}_x = -T^{r*}_y$.
‡ Limiting values calculated by O'Neill (1968).

TABLE 2. Forces and torques on a sphere near a wall

For the latter $\mathbf{F} = \mathbf{T} = 0$, from which we obtain

$$U_x = U_z = \Omega_y = 0,$$

$$U_y = \kappa h \frac{(a/2h)\, F_y^{r*} T_x^{s*} - F_y^{s*} T_x^{r*}}{F_y^{t*} T_x^{r*} - F_y^{r*} T_x^{t*}},$$

$$\Omega_x = \tfrac{1}{2}\kappa \frac{(2h/a)\, F_y^{s*} T_x^{t*} - F_y^{t*} T_x^{s*}}{F_y^{t*} T_x^{r*} - F_y^{r*} T_x^{t*}},$$

and

$$\Omega_z = -(\omega h/l)\,(T_z^{s*}/T_z^{r*}).$$

Values of $U_y/\kappa h$ and $\Omega_x/\frac{1}{2}\kappa$ as functions of h/a have been reported by Goldman, Cox & Brenner (1967*b*).

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